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Congruences for the Burnside module

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Notation 1.
- G : finite group
- $S(G)$: the set of all subgroups of G and G-set by conjugation
- $\Phi(G)$: the conjugacy class set of G
- Π : partially ordered set and G acts on it preserving the partially order
- $\rho : \Pi \to S(G)$: an order preserving G-map

Notation 2.
For any $\alpha \in \Pi$,
- $\Pi_\alpha := \{\beta \in \Pi \mid \beta \geq \alpha\}$
- $G_\alpha := \{g \in G \mid g\alpha = \alpha\}$

Defnition 3.
A pair (Π, ρ) is called a **G-poset** if it is satisfying the following condition: for any $\alpha \in \Pi$,

$$\rho(\alpha) \lhd G_\alpha \text{ and } \rho : \Pi_\alpha \to S(G)_{\rho(\alpha)} \text{ is injective.}$$

Note that $S(G)_{\rho(\alpha)} = S(\rho(\alpha)) \subset S(G_\alpha)$ and $G_\alpha \subset G_{\rho(\alpha)} = N_G(\rho(\alpha))$, the normalizer of $\rho(\alpha)$ in G.

Defnition 4.
A G-poset (Π, ρ) is called **complete** if

$$\rho : \Pi_\alpha \to S(G)_{\rho(\alpha)} \text{ is bijective for all } \alpha \in \Pi.$$

Definition 5.
A finite G-CW-complex X with the base point q is called a **Π-complex** if it is equipped with a specified set $\{X_\alpha \mid \alpha \in \Pi\}$ of subcomplexes X_α of X, satisfying the following four conditions:
(i) $X_\alpha \ni q$
(ii) $gX_\alpha = X_{g\alpha}$ for $g \in G$, $\alpha \in \Pi$,

(iii) $X_\alpha \subseteqq X_\beta$ if $\alpha \leqq \beta$ in Π, and
(iv) for any $H \in S(G)$,

$$X^H = \bigvee_{\rho(\alpha)=H} X_\alpha \quad \text{(the wedge sum of } X_\alpha\text{'s).}$$

Example 6.
Let $\alpha \in \Pi$. The G-CW-complex $(G/\rho(\alpha))^+ (= G/\rho(\alpha) \coprod \{*\})$ is a Π-complex ;

$$(G/\rho(\alpha))^+_\beta = \{g\rho(\alpha) \mid g\alpha \leq \beta\} \coprod \{*\} \text{ for any } \beta \in \Pi.$$

$\Longrightarrow (G/\rho(\alpha))^+$ is a Π-complex.

Definition 7. ([7])
Let Z and W be Π-complexes.

$$Z \sim W \Longleftrightarrow \chi(Z_\alpha) = \chi(W_\alpha) \quad \text{for all } \alpha \in \Pi$$

The set

$$\Omega(G, \Pi) = \{[Z] \mid Z \text{ is a } \Pi\text{-complex}\}$$

is called **Burnside module** .

•

$$[Z] + [W] := [Z \vee W]$$

Remark that
$\Longrightarrow \Omega(G, \mathcal{F})$ is a finitely generated free abelian group ($\longleftarrow$ Proposition 2)
Notation 8.
• $S((G), \alpha) := \{K \in S(G) \mid (K/\rho(\alpha)) \in \Phi(G_\alpha/\rho(\alpha))$
and $K/\rho(\alpha)$ is cyclic$\}$
• $\bar{\chi}(X) = \chi(X) - 1$ for any space X

Theorem A.
Let α be an element of Π.
Then we have $\sum_{K \in S((G),\alpha)} \frac{|G_\alpha/\rho(\alpha)|}{|N_{G_\alpha/\rho(\alpha)}(K/\rho(\alpha))|} \cdot \phi(|K/\rho(\alpha)|) \cdot \bar{\chi}(X^K_\alpha) \equiv 0 \mod |G_\alpha/\rho(\alpha)|$, *where* $\phi(|K/\rho(\alpha)|)$ *is the number of generators of the cyclic group* $K/\rho(\alpha)$.

《Proof of Theorem A》
Let (Π, ρ) be a G-poset and G_α the isotropy subgroup at α. Given a Π-complex X, we see

the $G_\alpha/\rho(\alpha)$-CW-complex $X^{\rho(\alpha)}$ is equipped with a Π-complex structure as following: $(X^{\rho(\alpha)})_\alpha = X_\alpha^{\rho(\alpha)}$ for all $\alpha \in \Pi$. By our definition of the Π-complex, it can be shown that $X_\alpha^{\rho(\alpha)} = X_\alpha$ for all $\alpha \in$ Let $\chi(X)$ be the Euler characteristic of X, and $\bar{\chi}(X) = \chi(X)-1$ Note that a map $f : \mathcal{F}_c(G_\alpha/\rho(\alpha)) \to \mathbb{Z}$; $K/\rho(\alpha) \mapsto \bar{\chi}(X_\alpha^K)$ satisfies a Burnside relation. By Burnside's lemma [6, Lemma 4.1], we have the desired result.

Lem

Suppose that a G-poset (Π, ρ) is complete. Let α be an elemnet of Π and K a subgroup with $K \supset \rho(\alpha)$. For a Π-complex X, it holds that

$$\bar{\chi}(X_\alpha^K) = \sum_{\beta\in\Pi \text{ with } \rho(\beta)=K, \beta<\alpha} \bar{\chi}(X_\beta).$$

Theorem B.

If a G-poset (Π, ρ) is complete, one has
$Im(\bar{\chi} : \Omega(G, \Pi) \to \bigoplus_{\alpha\in\mathcal{A}} \mathbb{Z})$

$$= \{(x_\alpha) \in \bigoplus_{\alpha\in\mathcal{A}} \mathbb{Z} \mid \sum_{K\in S((G),\alpha)} \frac{|G_\alpha/\rho(\alpha)|}{|N_{G_\alpha/\rho(\alpha)}(K/\rho(\alpha))|} \cdot \phi(|K/\rho(\alpha)|) \cdot x_{\alpha,(K)} \equiv 0 \mod |G_\alpha/\rho(\alpha)|\},$$

where $x_{\alpha,(K)}$ is some integer such that

$$x_{\alpha,(K)} = \begin{cases} x_\alpha & (K = \rho(\alpha)) \\ \sum_\beta x_\beta & (K \neq \rho(\alpha),\ \beta \text{ is some element of } \Pi \text{ with} \\ & \rho(\beta) = K, \beta < \alpha). \end{cases}$$

《Outline of Proof》

First we use **S** for the right side, and **Im** for the left side in the equation of Theorem B. Let $\mathcal{A} = \{\alpha_1, \cdots, \alpha_m\}$. By [4, Lemma 1.80], we can arrange elements of $\mathcal{A}$ such that

$$\alpha_i \le \alpha_j \Longrightarrow i \le j.$$

Define a map $P_{\le k} : \bigoplus_{i=1}^{m} \mathbb{Z}_{\alpha_i} \to \bigoplus_{i=1}^{k} \mathbb{Z}_{\alpha_i}$ by k coordinate maps $p_i : \bigoplus_{i=1}^{m} \mathbb{Z}_{\alpha_i} \to \mathbb{Z}_{\alpha_i}$ such that

$$P_{\le k}(x) = (p_1(x), \cdots, p_k(x)),$$

where each $\mathbb{Z}_{\alpha_i}$ is a copy of $\mathbb{Z}$. Note that $\mathrm{S} \subset \bigoplus_{i=1}^{m} \mathbb{Z}_{\alpha_i}$. It will now suffice to prove that

$$P_{\le m}(\mathbf{S}) = P_{\le m}(\mathbf{Im}).$$

We proceed by induction on k.

• the case of $k = 1$

By Theorem A and the previous Lemma, we have that $P_{\leq 1}(\mathbf{Im}) = P_{\leq 1}(\mathbf{S})$.

• the case of $k = m$

Remark that $P_{\leq m}(\mathbf{Im}) \subset P_{\leq m}(\mathbf{S})$. ($\longleftarrow$ the previous Lemma) Suppose that $P_{\leq k-1}(\mathbf{S}) = P_{\leq k-1}(\mathbf{Im})$ We prove $P_{\leq k}(\mathbf{S}) \subset P_{\leq k}(\mathbf{Im})$

Suppose that $P_{\leq k-1}(\mathbf{S}) = P_{\leq k-1}(\mathbf{Im})$. Let $y = (y_{\alpha_1}, y_{\alpha_2}, \cdots, y_{\alpha_{k-1}}, y_{\alpha_k}, y_{\alpha_{k+1}}, \cdots, y_{\alpha_m})$ be an element of $\mathbf{S}$. By assumption, there exists an element

$$x = (x_{\alpha_1}, x_{\alpha_2}, \cdots, x_{\alpha_{k-1}}, x_{\alpha_k}, x_{\alpha_{k+1}}, \cdots, x_{\alpha_m}) \in \mathbf{Im}$$

such that $x_{\alpha_1} = y_{\alpha_1}, x_{\alpha_2} = y_{\alpha_2}, \cdots, x_{\alpha_{k-1}} = y_{\alpha_{k-1}}$. Then we have

$$z = y - x = (0, 0, \cdots, 0, y_{\alpha_k} - x_{\alpha_k}, y_{\alpha_{k+1}} - x_{\alpha_{k+1}}, \cdots, y_{\alpha_m} - x_{\alpha_m}) \in \mathbf{S}.$$

Here we let $z_{\alpha_i} = y_{\alpha_i} - x_{\alpha_i}$, and $n_{\alpha,K} = \dfrac{|G_\alpha/\rho(\alpha)|}{|N_{G_\alpha/\rho(\alpha)}(K/\rho(\alpha))|} \cdot \phi(|K/\rho(\alpha)|)$. Consider the case of $\alpha = \alpha_k$. Then we have

$$\sum_{K \in S((G),\alpha_k)} n_{\alpha_k,K} \cdot z_{\alpha_k,(K)} \equiv 0 \mod |G_{\alpha_k}/\rho(\alpha_k)|.$$

Observe that the coefficient $z_{\alpha_k,(K)}$ $(K \neq \rho(\alpha_k))$ is equal to $\sum_\beta z_\beta$, where β is some element of Π with $\rho(\beta) = K, \beta < \alpha_k$. Thus the above equation implies

$$n_{\alpha_k,\rho(\alpha_k)} \cdot z_{\alpha_k,(\rho(\alpha_k))} \equiv 0 \mod |G_{\alpha_k}/\rho(\alpha_k)|.$$

Note that $n_{\alpha_k,\rho(\alpha_k)} = \dfrac{|G_{\alpha_k}/\rho(\alpha_k)|}{|N_{G_{\alpha_k}/\rho(\alpha_k)}(\rho(\alpha_k)/\rho(\alpha_k))|} \cdot \phi(|\rho(\alpha_k)/\rho(\alpha_k)|) = 1$. That is,

$$z_{\alpha_k} \equiv 0 \mod |G_{\alpha_k}/\rho(\alpha_k)|.$$

On the other hand, we have

$$\bigoplus_{\alpha \in \mathcal{A}} \bar{\chi}([(G/\rho(\alpha_k))^+]) = (\bar{\chi}_\alpha([(G/\rho(\alpha_k))^+])_{\alpha \in \mathcal{A}} = (\overbrace{0, 0, \cdots, 0}^{k-1}, |G_{\alpha_k}/\rho(\alpha_k)|, \cdots).$$

Hence there exists an integer $a \in \mathbb{Z}$ such that

$$y - x - a(\bar{\chi}_\alpha((G/\rho(\alpha_k))^+)) = (\overbrace{0, 0, \cdots, 0, 0}^{k}, \cdots).$$

$$y = x + a(\bar{\chi}_\alpha((G/\rho(\alpha_k))^+)) + (\overbrace{0, 0, \cdots, 0, 0}^{k}, \cdots).$$

By induction, we see immediately that

$$P_{\leq k}(y) = P_{\leq k}(x + a(\bar{\chi}_\alpha((G/\rho(\alpha_k))^+)) \in P_{\leq k}(\mathbf{Im}).$$

Example 9

Let p be a prime number. We set $G = C_p$ (a cyclic group of order p). Since $S(G) = \{\{e\}, G\}$ (e is the unit element of G), and the G-action on $S(G)$ is trivial, a Burnside module $\Omega(G, S(G))$ is a free abelian group generated by $[(G/\{e\})^+]$, $[(G/G)^+]$. Clearly $\Phi(G) = \{\{e\}, G\}$.

First, consider the case of $\alpha = \{e\}$. Since $S((G), \alpha) = \{\{e\}, G\}$, we get

$$\frac{|G|}{|G|} \cdot 1 \cdot x_{\{e\},(\{e\})} + \frac{|G|}{|G|} \cdot (p-1) \cdot x_{\{e\},(G)} \equiv 0 \mod p.$$

That is,

$$x_{\{e\},(\{e\})} \equiv x_{\{e\},(G)} \mod p.$$

By Theorem 1.7, there exists a Π-complex X such that $\bar{\chi}(X_{\{e\}}) = x_{\{e\},(\{e\})}$ and $\bar{\chi}(X_G) = x_{\{e\},(G)}$. Thus we have

$$\bar{\chi}(X_{\{e\}}) \equiv \bar{\chi}(X_G) \mod p.$$

In particular, if X has a Π-complex structure as follows:

$$X_\alpha = \begin{cases} X & (\alpha = \{e\}) \\ X^G & (\alpha = G), \end{cases}$$

the previous expression implies

$$\chi(X) \equiv \chi(X^G) \mod p.$$

Next for $\alpha = G$, since $S((G), \alpha) = \{G\}$, we obtain

$$\frac{1}{1} \cdot 1 \cdot x_{G,(G)} \equiv 0 \mod 1.$$

Immediately,

$$x_{G,(G)} \equiv 0 \mod 1.$$

This equation is true for any integer, and so there is no relation for Π-complexes.

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